PHYSICAL REVIEW E VOLUME 53, NUMBER 1 JANUARY 1996

Polymer-induced drag reduction in turbulent flows

D. Thirumalai
Institute for Physical Science and Technology, University of Maryland, College Park, Maryland 20742

J. K. Bhattacharjee*
Department of Physics and Astronomy, University of Maryland, College Park, Maryland 20742

(Received 9 December 1994)

We present a hydrodynamic model for polymer-induced drag reduction in turbulent flows. Our theory uses the stirred fluid model for isotropic turbulence. The effect of a polymer additive on the velocity field of the fluid is taken into account to lowest order in the concentration of the solute. The resulting theory predicts that the presence of the polymer molecules leads to the usual enhancement of viscosity at small length scales, while at intermediate length scales the effective viscosity is decreased, provided the concentration of the polymer exceeds a minimum value. We also suggest the existence of a polymer turbulent flow induced length scale l_p. An explicit calculation of l_p is in excellent agreement with recent experiments. The theory also suggests that the phenomenon of drag reduction can exist in the presence of nonpolymer additives, provided that $c\text{Re}^2$, with c being the concentration of the solute and Re the Reynold's number, is large enough.

PACS number(s): 47.55.−t

I. INTRODUCTION

In this paper we provide a theory for the phenomenon of polymer-induced drag reduction in turbulent flows. It is known that if small amounts of flexible polymers are added to a turbulent flow, the drag is significantly reduced. Lumley [1] provided a theoretical analysis of this phenomenon based on the notion of an increase in viscosity due to the addition of polymer molecules. More precisely, Lumley argued that due to possible stretching of the polymer molecule (if the strain rate in the turbulent flow is large) the viscosity in the region beyond the viscous sublayer would increase. This effectively leads to a thickening of the viscous sublayer leading to a decrease in the velocity gradient at the wall. Consequently the Reynold stress at the wall decreases, thus leading to a reduction in drag. This basic hypothesis has been made quantitative by Ryskin [2]. In Lumley's theory [1] the presence of walls plays a major role. In fact he suggested that the phenomenon cannot be explained without involving the reduction of stress at the wall. However, the results of certain experiments [3] indicate that drag reduction can take place far away from the walls. This led de Gennes and Tabor to speculate on a different origin for drag reduction. In the de Gennes–Tabor [4] scenario the polymers in turbulent flow exhibit elastic properties even at very low concentrations. Consequently, they can store up some of the cascading energy of turbulence. This energy is thus not dissipated by viscosity, and this gives rise to an effective drag reduction. The de Gennes–Tabor scenario introduces an undetermined stretching exponent n, and predicts the existence of a critical concentration below which drag reduction cannot be observed.

In order to understand this phenomenon we have undertaken an entirely different approach. We provide a hydrodynamic theory for which systematic calculations are possible without invoking *ad hoc* assumptions. It should be emphasized, however, that our approach is not without limitations. These are outlined at the end of the paper. We have used the idea behind the pioneering work of Wang [5] and Rabin, Wong, and Freed [6] to set up a hydrodynamic theory for turbulent flow of polymer solutions. In an earlier communication [7], we showed that drag reduction arises due to a decrease in the scale dependent viscosity at intermediate length scales. In the present work, we provide the details behind that calculation, and in the process provide an explicit expression for the polymer-flow-induced length scale down to which drag reduction is effective. This expression has no adjustable parameters, and turns out to be in excellent agreement with the measurements of Tong *et al.* [8,9]. This is presumably slightly fortuitous in that the flow field is experimentally found to be (i) inhomogeneous and (ii) anisotropic. We ignore both these aspects to develop a tractable theory and to expose the salient physical features. The rest of the paper is organized as follows. In Sec. II, we briefly recall the de Gennes–Tabor scenario. In Sec. III, the model for turbulent flow of polymer solutions is introduced, and in Sec. IV the single loop results are obtained. A short summary is provided in Sec. V.

II. de GENNES–TABOR SCENARIO

In this section we give a brief description of the de Gennes–Tabor scenario that is relevant for our purposes. It is assumed that for the turbulent fluid in the presence of trace amounts of flexible polymer coils the usual Kol-

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*On leave from the Department of Physics, Indian Institute of Technology, Kanpur, India.

1063-651X/96/53(1)/546(6)/$06.00

mogorov picture of energy cascade continues to be valid. Accordingly, at each length scale r, the velocity scale is $U(r)$, so that

$$\frac{U^3(r)}{r}=\varepsilon\ , \tag{2.1}$$

where ε is the rate at which energy is injected into the system to maintain the turbulence. In the usual situation (i.e., without polymers), this cascading energy is ultimately dissipated by molecular viscosity at the Kolmogorov scale l_d. The idea behind de Gennes–Tabor drag reduction scenario is that the polymer stores up the cascading energy in the form of elastic energy at a length scale l_p ($\gg l_d$), so that the dissipative scales are not reached. This is a viscoelastic effect and can be operative only if time scales are such that the basic hydrodynamic frequency $U(r)/r$ is higher than the polymer relaxation frequency (the inverse of the Zimm relaxation time). Since from Eq. (2.1), $U(r)/r$ increases with decreasing scale r, there exists a scale L below which the polymer effects can become important. Clearly

$$\frac{U(L)}{L}=\frac{1}{T_p}\ , \tag{2.2}$$

where the Zimm relaxation time T_p is proportional to the cube of the radius of gyration R. Using the Flory estimate for $R \sim N^{3/5}$, with N being the number of monomer units, $T_p \sim N^{9/5}$.

The central assumption in the de Gennes–Tabor theory is that in a turbulent flow which is presumed to be a combination of longitudinal and simple shear, the polymer elongation is scale dependent and obeys a scaling law

$$\lambda(r)=(L/r)^n\ , \tag{2.3}$$

where n is an unknown exponent. For a laminar flow, $n=1$ in two dimensions and $n=2$ in three dimensions. The value of n was tentatively asserted to be between 1 and 2. One now makes use of a result due to Pincus [10], which says that the elastic energy associated with the stretching of the polymer is proportional to $\lambda^{5/2}$, i.e.,

$$F_{el}=G\lambda^{5/2}\ , \tag{2.4}$$

where G is the elastic constant which is proportional to the polymer concentration. A combination of Eqs. (2.4), (2.3), and (2.1) reveals that the elastic energy becomes larger at shorter length scales. The Reynolds stress $\rho U^2(r)$ becomes smaller, and below a characteristic scale l_p the elastic energy will dominate and the cascading energy will be transferred to elastic potential energy. The scale l_p is obtained from

$$G\lambda^{5/2}(l_p)=\rho U^2(l_p)\ . \tag{2.5}$$

Using Eqs. (2.3) and (2.1),

$$\frac{l_p}{L}=\left[\frac{G}{\rho^2 U^2(L)}\right]^{\eta}\ , \tag{2.6}$$

where

$$\eta=1\bigg/\left[\frac{5n}{2}+\frac{2}{3}\right]\ . \tag{2.7}$$

For the energy to be transferred to polymeric deformations, before being dissipated, we must have $l_p \gg l_d$, the Kolmogorov dissipative scale, which is given by

$$\frac{l_d}{L}=[\mathrm{Re}(L)]^{-3/4}\ . \tag{2.8}$$

The condition for polymeric drag reduction is therefore

$$l_p > l_d$$

or

$$G/\rho^2 U^2(L) > \mathrm{Re}^{-3/4\eta}\ . \tag{2.9}$$

Since G is proportional to the concentration, this sets a lower limit on the polymer concentration for drag reduction to be effective. Notice that this concentration is much smaller than the usual overlap concentration, so that the viscosity of the solution is essentially the same as that of the solvent.

III. MODEL

In this section we introduce a hydrodynamic model which combines the Langevin equation description of polymer solution of Wang [3] with the randomly stirred turbulent model of DeDominicis and Martin [11]. This hydrodynamic description allows a direct calculation of the scale dependent turbulent viscosity and will lead, as a byproduct, as we shall show in Sec. IV, to predictions for the exponent n introduced in Eq. (2.3), and the minimum concentration required for drag reduction. The description of incompressible hydrodynamic turbulence can be achieved with a fair degree of success by the randomly stirred fluid model in which the Navier-Stokes equation for the fluctuating velocity field becomes

$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=-\frac{\nabla P}{\rho}+\nu\nabla^2\mathbf{v}+\mathbf{f}\ , \tag{3.1}$$

$$\nabla\cdot\mathbf{v}=0\ , \tag{3.2}$$

where ν is the kinematic viscosity, f is a random external force with a Gaussian distribution and moments specified in momentum space by

$$\langle f_i(\mathbf{k},\omega)f_j(\mathbf{k}'',\omega'')\rangle=\frac{D_0}{k^{D-4+y}}P_{ij}\delta(\mathbf{k}+\mathbf{k}'')\delta(\omega+\omega'')\ . \tag{3.3}$$

In the above, P_{ij} is the projection operator

$$P_{ij}=\delta_{ij}-k_ik_j/k^2\ , \tag{3.4}$$

and y is an exponent which can vary. The Kolmogorov spectrum corresponds to $y=4$. In the presence of the polymers there will be an additional force due to the polymer motion. We consider this additional force within the Rouse model, i.e., by neglecting self-avoidance and backflow effects. If $\mathbf{r}_n(t)$ is the position coordinate of the polymer segment at the nth contour point at time t, then the elastic energy is clearly

$$F_{el}=\sum_n F_{n,n+1}=\sum_n \frac{3}{2}\frac{(\mathbf{r}_{n+1}-\mathbf{r}_n)^2}{a^2} , \tag{3.5}$$

and the force at point $\mathbf{r}_n$ obtained is $-\partial F_{el}/\partial \mathbf{r}_n$, which becomes $(3/2a^2)[\mathbf{r}_{n+1}-\mathbf{r}_n+\mathbf{r}_{n-1}-\mathbf{r}_n]=\frac{3}{2}(\partial^2\mathbf{r}_n/\partial n^2)$, where we assume that the contour point index varies continuously. The polymer dynamics is governed by

$$\frac{\partial \mathbf{r}_n}{\partial t}=\mathbf{u}(\mathbf{r}_n,t)+\Gamma_0\frac{\partial^2\mathbf{r}_n}{\partial n^2} , \tag{3.6}$$

where $\mathbf{u}(\mathbf{r}_n,t)$ is the flow field [containing both elongational (laminar flow) and fluctuating parts] which carries the polymer, and Γ_0 is proportional to the mobility. The additional force term $\mathbf{F}_p$ in Eq. (3.1) coming from the polymer additive is

$$\mathbf{F}_p(\mathbf{r},t)=\sum_{\alpha=1}^{n_p}\int_0^N dn\, \delta(\mathbf{r}-\mathbf{r}_n^\alpha(t))\frac{\partial^2\mathbf{r}_n^\alpha}{\partial n^2} , \tag{3.7}$$

where N is the number of the monomer units, and n_p is the number of polymer coils. We need to write $\mathbf{F}_p(\mathbf{r},t)$ in terms of $\mathbf{u}(\mathbf{r}_n,t)$, and hence one needs to solve for $\mathbf{r}_n$ from Eq. (3.6). For an underlying elongational flow this has been done by Wang by assuming that the polymer alters the solvent velocity field only by a small amount. The result of such a calculation yields

$$\mathbf{F}_p=-\Gamma_0 c\int_{-\infty}^{t} dt''\mathbf{u}(k,t'') \times\sum_{p=1}^{\infty} e^{-2(t-t'')/\tau_p}\sum_i \frac{k_i^2}{1-\gamma_{ii}\tau_p} , \tag{3.8}$$

where c is the polymer concentration, $\tau_p \alpha N^2/p^2$ is the relaxation time for the pth Rouse mode and γ_{ii} is the strain matrix with the nonzero elements as $-2\gamma_{11}=-2\gamma_{22}=\gamma_{33}=\gamma$ for elongational flow where γ is the strain rate. To zeroth order, we will assume $\tau_p\gamma_{ii} \ll 1$ and hence one can drop these factors compared to unity in Eq. (3.8), which yields the isotropic force

$$\mathbf{F}_p=-ck^2\int_{-\infty}^{t} dt''\mathbf{u}(k,t'')\sum_{p=1}^{\infty} e^{-2(t-t'')/\tau_p} . \tag{3.9}$$

The force is linear in the solvent velocity and hence easily separable into steady state and fluctuating parts. With this our equations of motion for the velocity fluctuations in a randomly stirred fluid with polymer additives becomes in momentum space

$$\dot v_j(\mathbf{k})+\nu_0 k^2 v_j(\mathbf{k}) =\frac{i}{2}\sum_{l,m} P_{jlmk}(\mathbf{k})v_l(\mathbf{k})v_m(\mathbf{k}-\mathbf{p})+\mathbf{F}_p+f_i , \tag{3.10}$$

where

$$P_{ijl}(k)=k_jP_{il}(k)+k_lP_{ij}(k) . \tag{3.11}$$

The projection operators P_{ij} have been defined in Eq. (3.4), the correlation of the random force f_i specified in Eq. (3.3), and the force coming from the polymer additive has been written down in Eq. (3.9). If we wish to retain the anisotropy, we have to use the polymer force given in Eq. (3.8).

Notice that because of the assumption $\gamma_{ii}\tau_p \ll 1$ for all p our model cannot describe the coil-stretch transition. Thus the effects described by our theory occur due to changes in the scale dependent effective viscosity. Our theory, therefore, is markedly different from that of de Gennes and Tabor in that viscoelastic effects play no role in our formulation. Despite this fundamental difference we are led to quite similar results for the onset of polymer-induced drag reduction in turbulent flows.

IV. RESULTS

We first recount the known facts about the solution of Eq. (3.11) when $\mathbf{F}_p=\mathbf{0}$ (i.e., no polymer additives). The nonlinear terms in this equation conserve the total energy, and hence can only transfer energy from one scale to another, giving rise to an effective scale dependence viscosity $\nu(k)$ [12–14]. The scale dependent viscosity can be calculated by any one of several different techniques, and is found to obey the scaling law

$$\nu(k)=\Gamma k^{-y/3} \tag{4.1}$$

for $k \ll k_d$, where $k_d=l_d^{-1}$ is the Kolmogorov scale which is the wave number at which the molecular viscosity ν_0 begins to dominate the scale dependent viscosity of Eq. (4.1). Clearly

$$\nu_0=\Gamma k_d^{-y/3} , \tag{4.2}$$

while y is a free parameter in the force correlation and is a candidate for development of perturbation theory (to calculate amplitudes like Γ). The requirement that in the Kolmogorov case, the energy transfer is scale independent (entire energy transferred from one scale to the next) requires $y=4$ for the situation of interest. The amplitude Γ is universal and can be calculated in perturbation theory or self-consistent mode coupling theory [12–14].

The central quantity to calculate is the Green's function. For the linear case (with $\mathbf{F}_p=0$), this is given by

$$G_0^{-1}(k,\omega)=-i\omega+\nu_0k^2 . \tag{4.3}$$

Including the nonlinear terms, the full Green's function $G(k,\omega)$ obeys the Dyson equation

$$G^{-1}(k,\omega)=G_0^{-1}(k,\omega)+\Sigma(k,\omega) , \tag{4.4}$$

where $\Sigma(k,\omega)$ is the self-energy (we have used the opposite sign from the conventional one for reasons which are obvious). Then

$$G^{-1}(k,\omega)=-i\omega+\nu_0k^2+\Sigma(k,\omega) , \tag{4.5}$$

and we see immediately that $\Sigma(k,\omega)/k^2$ is a momentum and frequency dependent effective viscosity. The calculation of $\Sigma(k,\omega)$ follows standard diagrammatic perturbation theory.

We now turn to the situation where the polymer additives are present. In this case, the zeroth order Green's function (i.e., linear theory) is given by

$$G_0^{-1}(k,\omega)=-i\omega+\nu_0k^2+ck^2\sum_{p=1}^{\infty}\frac{1}{-i\omega+2/\tau_p} . \tag{4.6}$$

The hydrodynamic pole corresponding to $\omega\tau_p \ll 1$, occurs at $\bar{\nu}_0 k^2$, where

$$\bar{\nu}_0=\nu_0+c\sum_{p=1}^{\infty}\frac{\tau_p}{2} \ . \tag{4.7}$$

This is the enhancement of the viscosity at small scales and is the basis for Lumley's drag reduction criterion. In the Lumley picture [1], the increase of the viscosity leads to a thickening of the viscous sublayer or more precisely the spatial extent of the buffer layer separating the viscous sublayer and the inertial sublayer. The reduced velocity gradient due to the increased thickness of the viscous sublayer results in the reduction in the drag.

Our point here will be that while the polymer additives increase the viscosity at small length scales, they decrease the effective viscosity at large length scales and if one is far from the walls, then the reduction of the effective scale dependent viscosity in turn leads to the reduction in the drag. This sort of scale dependence which leads to enhancement in one regime and decrease in another due to polymer additives had also been previously noted for vorticity by Rabin and Zielanska [15].

We now return to Eq. (4.6) and working to the lowest order in k^2 and the concentration c, write the zeroth order Green's function as

$$G_0(k,\omega)=\sum_{k=1}^{\infty}A_i/(-i\omega+\alpha_i) \ , \tag{4.8a}$$

where

$$\alpha_1=\bar{\nu}k^2 \ , \quad \alpha_{i+1}=\frac{2}{\tau_i} \quad (i=1,2,\ldots) \ , \tag{4.8b}$$

$$A_1=1 \ , \quad A_{i+1}=-\tfrac{1}{4}ck^2\tau_i^2 \quad (i=1,2,\ldots) \ . \tag{4.8c}$$

If we consider scales larger than that determined by the time criterion of Lumley, cf. Eq. (2.2), the first term in Eq. (4.8a) dominates. In this case the scaling law corresponding to the pure solvent holds, and the polymer enhanced molecular viscosity is renormalized to $\bar{\nu}_R=\bar{\nu}_0+\bar{\nu}(k)$, where $\bar{\nu}(k)=\Gamma k^{-y/3}$. Our concern is to pick up the correction to $\bar{\nu}(k)$ due to the addition of a small concentration of polymer. To this end, we need to use the full Green's function of Eq. (4.8a) and calculate $\bar{\nu}(k)$ for dilute polymer solutions to first order in the concentration c. Using the fully dressed Green's function $G(k,\omega)$ and the corresponding correlation function $c(k,\omega)$, we have the usual self-consistent single loop contribution to $\Sigma(k,\omega)$, which in D dimensions is given by

$$\Sigma(k,\omega)=\frac{2k^2}{D-1}\int\frac{d^Dp}{(2\pi)^D}\frac{d\omega''}{2\pi}b(\mathbf{k},\mathbf{p},\mathbf{q})C(\mathbf{p},\omega'')\times G(\mathbf{k}-\mathbf{p},\omega-\omega'') \ , \tag{4.9}$$

where

$$b(\mathbf{k},\mathbf{p},\mathbf{q})=\frac{(\mathbf{k}\cdot\mathbf{q})^3}{k^4q^2}-\frac{(\mathbf{p}\cdot\mathbf{q})(\mathbf{k}\cdot\mathbf{q})}{p^2k^2}+\left[\frac{D-3}{2}\right]\left[1-\frac{(\mathbf{p}\cdot\mathbf{k})^2}{p^2k^2}\right] \ , \tag{4.10}$$

with $\mathbf{p}+\mathbf{q}=\mathbf{k}$. Naive power counting in the inertial range, where one can drop the molecular viscosity, ensures that the momentum dependence of $\Sigma(k,\omega)$ has to be $k^{2-y/3}$, which corresponds to the scale dependent viscosity behaving as $k^{-y/3}$. We now want to evaluate the zero-frequency self-energy $\Sigma(k)$, correct to first order in the concentration. Accordingly we make the following simplification on the right hand side:

$$G(k,\omega)=\frac{1}{-i\omega+\Sigma_0(k)}-\frac{\Gamma}{4}\frac{ck^{2-y/3}}{\nu}\sum\frac{\tau_i^2}{-i\omega+2/\tau_i} \ , \tag{4.11}$$

$$C(k,\omega)=\frac{K}{k^{D-4+y}}\frac{1}{\Sigma_0(k)}[G(k,\omega)+G^*(k,\omega)] \ , \tag{4.12}$$

where K is a constant and $\Sigma_0(k)$ is the self-energy with $c=0$.

Inserting the above forms of $G(k,\omega)$ and $C(k,\omega)$ into Eq. (4.9) and carrying out the straightforward frequency integration, we find

$$\Sigma(k)=\frac{2k^2}{D-1}\int_{\mathbf{p}+\mathbf{q}=\mathbf{k}}\frac{d^Dp}{(2\pi)^D}K\frac{b(\mathbf{k},\mathbf{p},\mathbf{q})}{\Sigma(p)p^{D-4+y}}\left[\frac{1}{\Sigma_0(p)+\Sigma_0(q)}-\frac{\Gamma c}{4\nu_0}\sum_j\tau_j^2\left[\frac{p^{2-y/3}}{L_J(q)}+\frac{q^{2-y/3}}{L_j(p)}\right]\right] \ , \tag{4.13}$$

where

$$L_j(k)=\frac{2}{\tau_j}+\Sigma_0(k) \ , \tag{4.14}$$

and $b(\mathbf{k},\mathbf{p},\mathbf{q})$ is an angular factor.

If we set $c=0$, $\Sigma(k)=\Sigma_0(k)=\Gamma k^{2-y/3}$, consistent with the standard scaling solution. To find how the correction term in Eq. (4.13) scales, we note that we have worked in the small τ approximation. Hence at this stage the appropriate thing is to assume τ^{-1} is dominant, and then power counting yields that the second term scales as k^{6-y}. We thus have the structure

$$\Sigma(k)=\Gamma k^{2-y/3}[1-c\Gamma_1k^{4-2y/3}] \ , \tag{4.15}$$

where the amplitudes Γ and Γ_1 involve universal numbers which are to be obtained from the integrals in Eq. (4.13). Our interest at present is in the emergence of polymer-induced scales, and we will return to the issue of the universal numbers at the end. From Eq. (4.15), we obtain the polymer-induced momentum scale as

$$k_p=(c\Gamma_1)^{1/[(2y/3)-4]} \ . \tag{4.16}$$

The inertial Kolmogorov range obtains for $k \ll k_p$. The scale k_p will control the size of the inertial range as long as $k_p \ll k_d$, the dissipative scale of Eq. (4.2). Thus for the drag reduction to be valid, we require $k_p \ll k_d$, or alternatively

$$\left[\frac{1}{c\Gamma_1}\right]^{1/(4-2y/3)} < \left[\frac{\Gamma}{\nu_0}\right]^{3/y} . \tag{4.17}$$

For the Kolmogorov case of $y=4$, this leads to

$$c > c_m = \frac{\nu_0}{\Gamma\Gamma_1} , \tag{4.18}$$

showing the existence of a minimum concentration c_m, below which the cascading energy will be completely dissipated and one would observe full viscosity.

The spirit of the present calculation is identical to the mode coupling calculation for obtaining corrections to scaling terms in the theory of critical transport coefficients near a second order phase transition point. The fluctuation enhanced transport has to crossover to the molecular transport as one moves away from the critical point (can be achieved by either increasing the temperature or by increasing the wave number), and this passage is characterized by the correction to scaling terms. The mode coupling calculations of this effort in dynamical critical phenomena [16] agrees with the corresponding renormalization group calculations [17] and are strongly supported by experiments [17]. In addition, for the case when the polymer additive is absent ($c=0$) the mode coupling calculations reproduce in detail all the results obtained by renormalization group methods. Our calculations here closely follow these, and the second term in Eq. (4.15) is the first correction to scaling generated. The long wavelength divergent effective viscosity $[\equiv \Sigma(k)/k^2]$ in the inertial regime has to cross over to the molecular viscosity for $k > k_d$, the Kolmogorov scale in the absence of the polymer additive. In the presence of the polymer additive we find a cross over scale k_p [cf. Eq. (4.16)], and this scale will control the crossover so long as $k_p < k_d$. It must be noted that in Eq. (4.15) we have established the leading behavior of the full crossover function. The validity of Eq. (4.15) is only for $k < k_p$. To cover the entire range of k a complete crossover function will be necessary.

We now provide an explicit expression for the polymer-induced length scale l_p, which is k_p^{-1}. To obtain all dimensions correctly, we need a factor of kT/m with the concentration c, measured as parts per million where m is the average mass of the polymer molecule. We have ignored a possible weak N dependence arising from the ratio of polymer to fluid masses and the radius of gyration of the polymer. We also need to observe that according to Kolmogorov scaling $\Gamma = \Gamma_0 \varepsilon^{1/3}$, where ε is the rate at which energy is injected to maintain the turbulence, and Γ_0 is a dimensionless universal number of order of unity. Now, comparing Eqs. (4.13) and (4.15), we can write

$$l_p^{4/3} = \frac{ckT}{m}\frac{\tau^3}{\nu_0}\varepsilon^{2/3}\frac{\Gamma_0^2 I_0}{8} , \tag{4.19}$$

where τ is the typical polymer relaxation rate and I_0 is another dimensionless number. Γ_0 and I_0 are determined by the two integrals in Eq. (4.13). If L is the typical length dimension in the problem (cell size in the experiment), then the Reynold's number is (V is the typical velocity)

$$\mathrm{Re} = \frac{VL}{\nu_0} , \tag{4.20}$$

and the energy injected per unit time is given by

$$\varepsilon = V^3/L . \tag{4.21}$$

This leads to

$$\varepsilon = \nu^3 \mathrm{Re}^3/L^4 , \tag{4.22}$$

and hence

$$l_p = \left[\frac{\Gamma_0^2 I_0}{8}\right]^{3/4} c^{3/4} \left[\frac{kT}{m}\frac{\tau^2}{L^2}\right]^{3/4} \times \left[\frac{\tau\nu_0}{L^{2/3}}\right]^{3/2} \mathrm{Re}^{3/2} . \tag{4.23}$$

In terms of l and l_p, the term in brackets in Eq. (4.15) is $1-(l_p/l)^{4-2y/3}$ [which for $y=4$ is $1-(l_p/l)^{2/3}$], and thus l_p is the lowest length scale that our theory picks out. At this level of accuracy we thus find that the drag reduction is down to a length scale of l_p. Our theory is not valid for length scales less than l_p.

For the experiment of Tong and co-workers [8,9], where trace amounts of polyethylene oxide was used, $\mathrm{Re} \simeq 3\times 10^5$, $\tau = 2.5\times 10^{-2}$ s, $m = 8.3\times 10^{-18}$ g, $L = 2.8$ cm, and $\nu_0 = 10^{-2}$ cm^2/s. We obtain $l_p \simeq 0.1$ cm from Eq. (4.21), with $\Gamma_0^2 I_0$ set equal to 1.5 as obtained from the evaluation of the integrals. This is in excellent (almost exact) agreement with the data, as we stated earlier. This fortuitous agreement suggests that the theory developed here may be of at least limited ability in analyzing experiments involving the interactions between polymers and turbulent flow.

Returning to Eq. (4.18), we now see that for a given ε the dependence of c_m on ε is ε^{-1}, and this allows us to determine the exponent n introduced by de Gennes and Tabor. If we look at their ε dependence of the critical concentration, then it is clear from Eq. (2.9) that $c \sim \varepsilon^{1-3/4\eta}$, and hence we obtain $3/4\eta = 2$ or

$$n = \tfrac{4}{5} . \tag{4.24}$$

This is slightly smaller than the exponent conjectured by de Gennes and Tabor [4].

V. CONCLUSIONS

In this paper we have elaborated on a hydrodynamic theory of polymer-induced drag reduction introduced a few years ago [7]. The major advantage of our formulation is that once the model for isotropic turbulence is accepted the results follow without recourse to additional *ad hoc* assumptions. The results show that flexible polymer molecules lead to an enhancement of molecular viscosity at small length scales, while at intermediate length scales the effective viscosity is decreased. If the polymer-induced length scale l_p exceeds the intrinsic Kolmogorov dissipative length, then the reduction in viscosity down to l_p leads to a natural explanation for drag reduction. This restriction in turn leads to a minimum value of the concentration of polymers below which polymer-induced drag reduction becomes ineffective. It is in fact remarkable that this simple model gives an explicit expression for l_p, which for the parameters used in the experiments yields a value in agreement with the measured result. Our theory can be further tested by measuring l_p as a function of c and Re. According to Eq. (4.23), l_p should scale as $(c\,\mathrm{Re}^2)^{3/4}$. This, in principle, can be verified experimentally.

The theory presented here also suggests that drag reduction can be caused by adding other substances as long as the molecules possess a spectrum of relaxation times. In particular the calculations would suggest that if the Rouse-mode sum in Eq. (3.10) is replaced by a single term, then one would obtain an explicit expression for the solute flow induced length l_p. However, if we insist that this has to exceed l_k for drag reduction to be operative, then the associated concentration of the solute has to be so large as to destroy the underlying turbulent flow. Thus we expect that flexible linear polymer molecules are perhaps the most efficient drag reducers. This can be seen by examining the N dependence of c_m. It is easy to show from Eq. (4.17) that

$$c_m \sim N^{-x}\varepsilon^{-1}\,, \tag{5.1}$$

where $x=4$ for Rouse chains and $x \simeq 3.6$ for the Zimm model. Thus only very small amount of polymer molecules need be added if N is very large. On the other hand, if N were relatively small then the concentration of the additive has to be sufficiently large. At such high concentrations the flow may no longer be turbulent, and hence drag reduction may not be practical. The same conclusion is reached by noting that the critical parameter controlling the efficiency of drag reduction is $(c\,\mathrm{Re}^2)^{3/2}$. Thus if c is large then the kinematic viscosity increases, implying that a substantial increase in ε is needed to satisfy the inequality $l > l_d$.

The theory described here should only be viewed as a first step toward a systematic calculation of the interaction between polymer additives and solvent velocity fields in a turbulent flow. In the calculations we had ignored terms on the order of $\gamma_{ii}\tau_p$ [see Eqs. (3.8) and (3.9)]. If we expand the term $1/(1-\gamma_{ii}\tau_p)$ in Eq. (3.8) to order $\gamma_{ii}\tau_p$, then it is clear that all conclusions of our calculations remain unchanged. It would be of interest to extend these calculations for arbitrary values of the strain rate, so that the effect of flow-induced stretching of the chain can be calculated. We should emphasize that although our calculations do not explicitly address the chain stretching effect, the pressure of minimum concentration seems to lead implicitly to the notion that the chain is locally stretched. This is most clearly seen by comparing our theory to the predictions of de Gennes and Tabor [see Eq. (4.24)].

Despite the fundamental differences between our theoretical description and the de Gennes–Tabor picture of polymer drag reduction, there are qualitative similarities. The connection between the two is not clear to us. Although we have provided a reasonable starting point to describe the drag reduction phenomenon, many problems remain. The assumption of isotropic turbulence certainly needs to be relaxed. In addition it would be useful to calculate the scale dependent viscosity for different underlying flows in order to check the generality of the origin of drag reduction. Finally a careful analysis of the presence of walls needs to be done to compare with the traditional Lumley picture that may be relevant to drag reduction induced by rodlike molecules. We hope to return to some of these problems in the future.

ACKNOWLEDGMENT

This work was supported in part by a grant from the National Science Foundation.

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